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ON PROBABILISTIC INTERPRETATION OF FUZZY NUMBERS

The paper is devoted to probabilistic interpretation of fuzzy numbers. The relationships between the fuzzy numbers and the interval random set are studied. The ordering of fuzzy numbers based on the stochastic dominance and the defuzzification problem based on the functional representation of preference relation and random simulation are presented. The problems of approximation of the fuzzy numbers by the crisp intervals and trapezoidal fuzzy numbers are investigated.

1. Introduction

This paper is devoted to some problems connected with relationships between the fuzzy numbers and the interval random sets. Generally, we cannot use the probabilistic interpretation of the value of membership function of the fuzzy set A. Usually, we treat the value $\mu_A(x)$ as the possibility that x belongs to A. It is a degree of membership of this point in a fuzzy set. But some authors [6]–[8] use probability interpretation in specific situations. The relationships between the fuzzy sets and the random sets enable them to do it. They interpret the degree of membership $\mu_A(x)$ as probability that the point x belongs to some random set **S**.

In section 2, the interval random sets and the relationships between the fuzzy numbers and these random sets are investigated. The ordering of fuzzy numbers based on the stochastic dominance is studied in section 3. Section 4 is devoted to an order defined on the trapezoidal fuzzy numbers, the special case of fuzzy numbers with the linear membership function. The defuzzification of fuzzy numbers based on the functional representation of order relations and the random simulation is discussed in section 5. In section 6, we study the approximation of fuzzy number by

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other fuzzy numbers with simpler membership functions – the intervals and trapezoidal fuzzy numbers.

2. Interval random sets

First, we recall some definitions and notions connected with fuzzy numbers. The *fuzzy number A* is a normal and convex fuzzy subset of real line with upper semi-continuous membership function μ_A: $R \to [0, 1]$. Every r-level $A_r = \{x: \mu_A(x) \geq r\}$, where $r \in (0, 1]$, of such a fuzzy set is a closed interval in this case. We can describe the membership function of the fuzzy number A as follows:

$$\mu_A(x) = \begin{cases} 0 & x < a_1 \text{ or } a_4 < x \\ f_A(x) & a_1 \leq x < a_2 \\ 1 & a_2 \leq x \leq a_3 \\ g_A(x) & a_3 < x \leq a_4 \end{cases} .$$

The functions f_A and g_A are called the left and right sides of the fuzzy number A [8]. For the sake of simplicity, we assume in this paper that these functions are continuous and strictly monotone. The function f_A is increasing and g_A is decreasing. Moreover, we assume that support of the fuzzy number A, i.e., the set $A_0 = \overline{\{x : \mu_0(x) > 0\}}$, where $\overline{D}$ is the closure of the crisp set D, is the bounded interval $[a_1, a_4]$; generally it can be unbounded.

The fuzzy number with linear sides is called a trapezoidal fuzzy number. We denote this fuzzy set by $T(a_1, a_2, a_3, a_4)$. For $a_2 = a_3$ we obtain a triangular fuzzy number.

The *random set* **S** is a measurable mapping from probability space $(\Omega, \mathfrak{I}, \Pr)$ to some class of the subsets of space X. In our situation, we assume that space X is a real line R and this class is a family of all closed intervals of real line., i.e., $\mathbf{S}(\omega) = [S_1(\omega), S_2(\omega)]$ [8]. This random set is called the *interval random set*. We can interpret every interval random set as a random variable taking value in $\mathbb{IR}^2 = \{(x,y): x, y \in R, x \leq y\}$ or as a pair of two random variables S_1, S_2.

The fuzzy number A induces a class of random sets satisfying condition $\Pr(x \in \mathbf{S}) = \mu_A(x)$. We can treat the value of membership function $\mu_A(x)$ as the probability that element x belongs to the random set **S**. We distinguish a random set $S_A = G_A)U$ generated by the level multifunction $G_A(r) = \{x: \mu_A(x) \geq r\}$ and a random variable U defined on the unit interval. When random variable U is uniformly distributed on [0, 1] we obtain a *consonant random set* generated by a fuzzy number A. Such a random set satisfies the condition $\Pr(x \in \mathbf{S}) = \mu_A(x)$. For other random variables that are not uniformly distributed, this formula is not satisfied.

The random variables $S_{A,1}$, $A_{A,2}$ induced by the consonant random set $\mathbf{S}_A$ are equal to $S_{A,1} = f_A^{-1} \circ U$ and $S_{A,2} = g_A^{-1} \circ U$, respectively. The expected value of this interval random set is called an *expected interval of fuzzy number A* [5], [8], i.e.,

$$EI(A) = [E(S_{A,1}), E(S_{A,2})] = \left[a_2 - \int_{a_1}^{a_2} f_A(x)dx \, , \, a_3 + \int_{a_3}^{a_4} g_A(x)dx \right] = \left[\int_0^1 f_A^{-1}(t)dt \, , \, \int_0^1 g_A^{-1}(t)dt \right].$$

It does not depend on the member of the class of interval random sets generated by fuzzy number A [8]. The center of such an interval is called an *expected value* of A and is denoted by $EV(A)$, i.e.,

$$EV(A) = \frac{1}{2}(E(S_{A,1}) + E(S_{A,2})),$$

and the ends of this interval are called the expected lower and upper values. The expected interval and expected value of trapezoidal fuzzy number A equal

$$EI(A) = [(a_1 + a_2)/2, (a_3 + a_4)/2],$$

$$EV(A) = (a_1 + a_2 + a_3 + a_4)/4.$$

3. Natural ordering of fuzzy numbers based on the stochastic dominance

There are many methods of ordering fuzzy numbers [1], [9], [11], [14]. We investigate the methods based on the random set representation of such fuzzy sets. First, we introduce the natural order $\preceq_1$:

$$A \preceq_1 B \;\Leftrightarrow\; f_A^*(x) \geq f_B^*(x) \text{ and } g_A^*(x) \leq g_B^*(x),$$

where A, B are the fuzzy numbers and

$$f_A^*(x) = \begin{cases} 0 & x < a_1 \\ f_A(x) & a_1 \le x < a_2 \\ 1 & a_2 \le x \end{cases}, \qquad g_A^*(x) = \begin{cases} 1 & x \le a_3 \\ g_A(x) & a_3 < x \le a_4 \\ 0 & a_4 \le x \end{cases}.$$

It is easy to see that it is the partial order. There exists a class of noncomparable fuzzy numbers in this order relation, e.g., when $a_1 < b_1 < b_4 < a_4$ or $a_1 < b_1 < b_2 < a_2$.

Now, we will study the connection between this natural relation $\preceq_1$ and the stochastic dominance. First, we will investigate the left sides f of the fuzzy numbers only and assume, for simplicity, that $a_1 > 0$. The cumulative distribution function of the random variable $S_{A,1}$ generated by the consonant random set $\mathbf{S}_A$ is equal to the left side extension $f_A^*(x)$ of this fuzzy number [8]. We know [11], [13] that a random variable Y *stochastically dominates* a random variable X ($X \le_{\mathrm{st}} Y$) if $F_X(x) \ge F_Y(x)$ for all $x \ge 0$, where $F_X(x)$, $F_Y(x)$ are the cumulative distribution functions of random variables X and Y. This fact implies that if $f_A^*(x) \ge f_B^*(x)$, then $S_{A,1} \le_{\mathrm{st}} S_{B,1}$. On the other hand, when we have the following relation between the right sides of the fuzzy numbers A and B: $g_A^*(x) \le g_B^*(x)$, then $S_{A,2} \le_{\mathrm{st}} S_{B,2}$. Therefore, $1 - g_A^*(x)$ is the cumulative distribution function of the random variable $S_{A,2}$. We arrive at the following conclusion.

Theorem 1. The natural order $\preceq_1$ between the fuzzy numbers is the intersection of two stochastic dominances between the sides of these fuzzy numbers, i.e.,

$$A \preceq_1 B \iff S_{A,1} \le_{\mathrm{st}} S_{B,1} \text{ and } S_{A,2} \le_{\mathrm{st}} S_{B,2}.$$

We can construct, in a similar way, a family of relations ordering the fuzzy numbers based on the higher degree stochastic orders. First, we recall the definition of the n-degree stochastic order [11], [13]. Denote ${}^1G_X(x) = 1 - F_X(x)$ and

$${}^{n+1}G_X(x) = \int_x^\infty {}^nG_X(t)dt \qquad \text{for } n = 1, 2, \ldots,$$

where X a random variable and $x \ge 0$. A random variable Y dominates a random variable X in n-degree stochastic order ($X \le_{\mathrm{s},n} Y$) if $E(X^k) \le E(Y^k)$ for $k = 1, 2, \ldots, n - 1$ and ${}^nG_X(x) \le {}^nG_Y(x)$ for all $x \ge 0$. When $E(X^k) \le E(Y^k)$ for all k, we say that Y dominates X in stochastic order of degree infinity ($X \le_{\mathrm{s},\infty} Y$).

We can define an *n-degree natural order* $\preceq_n$ between fuzzy numbers A and B, using the above stochastic orders:

$$A \preceq_n B \iff S_{A,1} \le_{\mathrm{s},n} S_{B,1} \text{ and } S_{A,2} \le_{\mathrm{s},n} S_{B,2}.$$

These ordering relations are the partial orders. The n-degree natural order is a weaker relation than $(n-1)$-degree natural order. If we use n-degree natural order, then many more pairs of fuzzy numbers can be ordered.

The exponential order $\le_e$ defined as

$$X \le_e Y \iff E(e^{\alpha X}) \le E(e^{\alpha Y})$$

for all $\alpha > 0$, is a weaker order than $\le_{s,n}$ for any n.

It is easy to see that $E(X) = \int_0^\infty (1 - F_X(t))dt$, where $X \ge 0$, then we can define the second degree natural order in the following way

$$A \preceq_2 B \iff \int_x^\infty (1 - f_A^*(t))dt \le \int_x^\infty (1 - f_B^*(t))dt \text{ and } \int_x^\infty g_A^*(t))dt \le \int_x^\infty g_B^*(t))dt$$

for any $x \ge 0$.

4. Trapezoidal fuzzy numbers

The trapezoidal fuzzy numbers are the simplest case of fuzzy numbers. Their sides are linear. In this situation, the natural order is defined by the following formula:

$$A \preceq_1 B \iff a_1 \le b_1, a_2 \le b_2, a_3 \le b_3, a_4 \le b_4.$$

We see that the relation between two trapezoidal fuzzy numbers depends on the parameters a_i and b_i, where $i = 1,2,3,4$, only. This is a simpler product order on $\mathrm{IR}^4 = \{(x_1, x_2, x_3, x_4): x_i \in R, x_1 \le x_2 \le x_3 \le x_4)$.

Example 1. Let $A = T(3, 5, 6, 7)$, $B = T(4, 6, 8, 9)$ and $C = T(2, 8, 9, 10)$. We obtain the relation A]$_1$ B, but the pair of fuzzy numbers A and C are not in this relation.

Now, we investigate the second degree natural order]$_2$ between the trapezoidal fuzzy numbers. First, we study the left sides f of them. Let $\pi_A(x) = \int_x^{a_2} (1 - f_A(t))dt$.

Then we obtain

$$\pi_A(x) = \begin{cases} (a_1 + a_2)/2 - x & 0 \le x < a_1 \\ \dfrac{(a_2 - x)^2}{2(a_2 - a_1)} & a_1 \le x < a_2 \\ 0 & a_2 \le x \end{cases}.$$

We know from the definition of the second degree of stochastic order that

$$S_{1,A} \leq_{s,2} S_{1,B} \quad \Leftrightarrow \quad \pi_A(x) \leq \pi_B(x)$$

for any $x \geq 0$. If a trapezoidal fuzzy number B dominates a trapezoidal fuzzy number A in the second degree natural order, i.e., $A \preceq_2 B$, then the parameters of such fuzzy numbers satisfy two conditions: $a_1 + a_2 \leq b_1 + b_2$ and $a_2 \leq b_2$. Therefore, $\pi_A(0) = (a_1 + a_2)/2 \leq \pi_B(0) = (b_1 + b_2)/2$, $\pi_A(a_2) = 0 \leq \pi_B(a_2)$ and $\pi_B(x)$ is a continuous, decreasing function. On the other hand, these two conditions imply that $\pi_A(x) \leq \pi_B(x)$ for any $x \geq 0$. This fact emerges from the properties of functions $\pi_A(x)$ and $\pi_B(x)$. The results for right sides g are similar to those for left sides.

Theorem 2. Let A, B be fuzzy numbers, then

$$A \preceq_2 B \quad \Leftrightarrow \quad a_1 + a_2 \leq b_1 + b_2,\ a_2 \leq b_2 \text{ and } a_3 + a_4 \leq b_3 + b_4,\ a_4 \leq b_4. \tag{1}$$

The order $\preceq_1$ is a stronger relation than order $\preceq_2$, then there exist pairs of trapezoidal fuzzy numbers, which satisfy relation $\preceq_2$, but do not satisfy relation $\preceq_1$.

Example 2. We have $A \preceq_2 C$, where A and C are from example 1.

We can show in a similar way that the higher degree natural orders $\preceq_n$, where $n > 2$, and exponential order $\preceq_e$ are characterized by the same two conditions (1). Then, it is sufficient to use orders $\preceq_1$ and $\preceq_2$ only for the trapezoidal fuzzy numbers.

Example 3. Let $A = T(3, 8, 10,12)$ and $B = T(1, 9, 10,12)$. If $A \preceq_e B$ then $E(e^{\alpha S_{A,1}}) \leq E(e^{\alpha S_{B,1}})$ for all $\alpha > 0$. Let $\varphi_A(\alpha) = E(e^{\alpha S_{A,1}}) = \dfrac{e^{\alpha a_2} - e^{\alpha a_1}}{\alpha(a_2 - a_1)}$. But $\varphi_A(0.2) = 3.131 > \varphi_B(0.2) = 3.018$ and $\varphi_B(0.5) = 20.05 < \varphi_B(0.5) = 20.09$. These fuzzy numbers do not satisfy the relation of the exponential order.

5. Defuzzification

The method of ranking fuzzy numbers presented in the previous sections generates a partial order. There is some class of uncomparable fuzzy numbers in these cases. Now, we present some examples of the weak orders of the fuzzy numbers, i.e., the complete and transitive relations. The completeness ensures that every pair of the fuzzy numbers is comparable in this relation.

The more natural relation is the order $]_E$ generated by the expected values of fuzzy numbers [9], i.e.,

$$A\ ]_E\ B \iff EV(A) \le EV(B).$$

Example 4. Let $A = T(3, 8, 10,12)$ and $B = T(1, 9, 10,12)$, then $EV(A) = 8.25$ and $EV(B) = 8$. We see that $A\ ]_E\ B$ in this case, but they are uncomparable in the exponential order.

We can generalize this order introducing a relation based on the expected interval $EI(A)$ of the fuzzy number. This order, denoted by symbol $]_\lambda$, is generated by a convex combination of the expected lower and upper values, i.e.,

$$A\ ]_\lambda\ B \iff \lambda E(S_{A,1}) + (1-\lambda)E(S_{A,2}) \le \lambda E(S_{A,1}) + (1-\lambda)E(S_{B,2}), \tag{2}$$

where $0 \le \lambda \le 1$. The relation $]_\lambda$ is the weak order, too.

In the decision making problems under fuzzy environment we must compare the expected fuzzy utilities or payoffs. The value of the coefficient λ reflects the tendency of the decision maker towards risk and assurance in this situation. We can also model his her tendency to value the membership function, the degree of possibility, by some distortion function φ. It is an increasing function φ: $[0, 1] \to [0, 1]$, satisfying condition: $\varphi(0) = 0$, $\varphi(1) = 1$. We replace the expected values of the random variables $E(S_{A,i})$, where $i = 1, 2$, with expected values with respect to the distortion probabilities $E_\varphi(S_{A,i})$ in this case, where

$$E_\varphi(S_{A,1}) = \int_0^\infty \varphi(1 - f_A(x))dx = \int_0^1 (1 - f_A)^{-1}(r)d\varphi(r),$$

$$E_\varphi(S_{A,2}) = \int_0^\infty \varphi(f_A(x))dx = \int_0^1 g_A^{-1}(r)d\varphi(r).$$

These integrals are strictly connected with Yaary's dual theory [15], which is an alternative to the classical utility theory. We denote the order between fuzzy numbers generated by the convex combination of modified expected values $E_\varphi(S_{A,i})$ by symbol $]_{\lambda,\varphi}$.

Let $C = A + B$ be the arithmetic sum of fuzzy numbers A and B, then [8]

$$f_C^{-1}(r) = f_A^{-1}(r) + f_B^{-1}(r),$$

$$g_C^{-1}(r) = g_A^{-1}(r) + g_B^{-1}(r).$$

We see, using the above properties, that the relation $]_{\lambda,\varphi}$ is consistent with the arithmetical sum of fuzzy numbers.

The expected value and the convex combination of the expected lower and upper values play the role of *functional representations* $V(A)$ of the above weak orders, i.e., there exists a real function V defined on the class of fuzzy numbers which satisfies the following condition:

$$A \preceq B \iff V(A) \le V(B).$$

On the other hand, every such functional V defined on the class of fuzzy numbers induces an order relation on this class.

Using this functional representation V, we can generate a real number $V(A)$ for every fuzzy number A. This is an example of the more general problem that is, defuzzification, where the fuzzy number is replaced by the real number. We can apply defuzzification not only to the ordering of fuzzy numbers, but it can be used in the fuzzy control and decision making problems [16], [9].

Using simulation methods based on the random set representation of fuzzy numbers we obtain other defuzzification methods. We present a two-step procedure [4, 3]. First, we generate a random set $\mathbf{S}_A$ induced by the fuzzy number A and second, we generate the number x_A from this subset. In the case of the consonant interval random set $\mathbf{S}_A = G_A \circ U$ this procedure is equivalent to the following two-step simulation:

Step 1: Generate a value r of the uniform random variable U over (0, 1].

Step 2: Generate a value x_A of some random variable Z over the level set $G_A(r)$.

We assume that these random variables are independent.

With Z being a uniform random variable we obtain a simpler procedure of generating a real number from the random variable [4]

$$Q = f_A^{-1}(U) + T(g_A^{-1}(U) - f_A^{-1}(U)),$$

where T and U are independent uniform random variables over [0, 1] and (0, 1]. The expected value of this random variable is equal to the expected value of the fuzzy set A: [4], [8]

$$E(Q) = \frac{1}{2}\int_0^1 (f_A^{-1}(t) + g_A^{-1}(t))dt = EV(A).$$

If we want to model the proclivity to risk in decision-making problems, we can use other than the uniform random variable Z. For instance, we can apply the triangular distribution function Z with the following density function

$$\rho_r(x) = \begin{cases} \dfrac{2(x-s_1)}{\lambda(s_2-s_1)^2} & s_1 \le x \le (1-\lambda)s_1 + \lambda s_2 \\ \dfrac{2(s_2-x)}{(1-\lambda)(s_2-s_1)^2} & (1-\lambda)s_1 + \lambda s_2 < x \le s_2 \end{cases},$$

where $G_A(r) = [s_1, s_2]$ and $\lambda \in [0, 1]$ is an optimism index. The pessimistic, risk averse decision maker chooses the parameter $\lambda = 0$ and the optimistic one, i.e., the risk lover takes $\lambda = 1$. The values of λ between 0 and 1 correspond with intermediate proclivity to risk. The cumulative distribution function is

$$F_r(x) = \begin{cases} \dfrac{1}{\lambda}\left(\dfrac{x-s_1}{s_2-s_1}\right)^2 & s_1 \le x \le (1-\lambda)s_1 + \lambda s_2 \\ 1 - \dfrac{1}{1-\lambda}\left(\dfrac{s_2-x}{s_2-s_1}\right)^2 & (1-\lambda)s_1 + \lambda s_2 < x \le s_2 \end{cases}.$$

Let $0 \le t \le \lambda$. Then $x = s_1 + \sqrt{\lambda t}\,(s_2 - s_1)$ is a solution of equation $t = F_r(x)$. For $\lambda < t \le 1$, we obtain $x = s_2 - \sqrt{(1-\lambda)(1-t)}\,(s_2 - s_1)$. Using the above investigation we can generate in this case, a single value from conditional random variable

$$Q = \begin{cases} f_A^{-1}(U) + \sqrt{\lambda T}\,(g_A^{-1}(U) - f_A^{-1}(U)) & 0 \le T \le \lambda \\ g_A^{-1}(U) - \sqrt{(1-\lambda)(1-T)}\,(g_A^{-1}(U) - f_A^{-1}(U)) & \lambda < T \le 1 \end{cases},$$

where T and U are independent uniform random variables over [0, 1] and (0, 1]. The expected value of this random variable is

$$E(Q) = (\lambda - \frac{2}{3}(\lambda^{7/2} - \sqrt{1-\lambda}\,(1-\lambda^3)))E(S_{A,1})$$
$$+ (1 - \lambda + \frac{2}{3}(\lambda^{7/2} - \sqrt{1-\lambda}\,(1-\lambda^3)))E(S_{A,2}).$$

For a risk averse decision maker, $\lambda = 0$, we obtain $E(Q) = \frac{2}{3}E(S_{A,1}) + \frac{1}{3}E(S_{A,2})$ and for a risk lover, $\lambda = 1$, we have $E(Q) = \frac{1}{3}E(S_{A,1}) + \frac{2}{3}E(S_{A,2})$. The expected value $E(Q)$ is equal to the expected value of fuzzy number $EV(A)$ for a neutral decision maker, i.e., when $\lambda = 0.5$.

6. Approximation of fuzzy number

Now, we study the problem of approximation of a fuzzy number by another fuzzy number with a simpler membership function. The fuzzification is an extreme example of such a problem. We approximate the fuzzy number by the real number in this case.

Such approximation done by the proper interval is another example of this problem. To that end we can use the expected interval. Then every fuzzy number A is represented by the interval $EI(A)$. The width $w(E(A))$ of the expected interval is equal to the width $w(A)$ of this fuzzy number, i.e., [2]

$$w(A) = \int_{-\infty}^{\infty} \mu_A(x)dx = E(S_{A,2}) - E(S_{A,1}) = w(E(A)).$$

Chanas in [2] introduced another interval approximation of fuzzy number. He constructed the interval I_0 of the same width as the fuzzy number A, which is located nearest to this fuzzy number with respect to the Hamming distance, i.e.,

1) $w(I_0) = w(A)$,

2) $H(I_0, A) = \min_I H(I, A)$

where I is an interval satisfying equality 1) and $H(I, A)$ is the Hamming distance between I and A:

$$H(I, A) = \int_{-\infty}^{\infty} |\mu_I(x) - \mu_A(x)| dx,$$

where $\mu_I(x)$ is a characteristic function of interval I. The first point z_0 of this optimal interval satisfies the following condition [2]

$$f_A(z_0) = g_A(z_0 + w(A)). \tag{3}$$

Chanas showed that for fuzzy numbers of L-R type, when the left shape function L is equal to the right shape function R, and only for such a fuzzy number, this optimal approximation interval is the expected interval, i.e., $I_0 = EI(A)$. The membership function of a fuzzy number A of L-R type has the following form:

$$\mu_A(x) = \begin{cases} L\left(\dfrac{a_2 - x}{\alpha_A}\right) & x < a_2 \\ 1 & a_2 \le x \le a_3 \\ R\left(\dfrac{x - a_3}{\beta_A}\right) & a_3 < x \end{cases},$$

where L and R are continuous, non-increasing functions, defined on $[0, +\infty)$, strictly decreasing to 0, when they are positive and fulfilling the conditions $L(0) = R(0) = 1$, $\alpha_A, \beta_A > 0$.

The above property, i.e., $I_0 = EI(A)$, is satisfied for the general symmetric fuzzy numbers, too. For this case, $g_A(x) = f_A(a_1 + a_4 - x)$. Using equation (3) and the symmetry of A, we obtain

$$z_0 = \frac{a_1 + a_4 - w(A)}{2} = E(S_{A,1}).$$

Then $I_0 = EI(A)$.

We may obtain the interval with a smaller Hamming distance to fuzzy number A than Chanas optimal interval I_0, when we remove the assumption about the equality of width, i.e., $w(I_0) = w(A)$.

Example 5. Let A be a fuzzy number with the following sides:

$$f_A(x) = 1 - \frac{1}{4}(x-2)^2 \qquad 0 \le x \le 2,$$

$$g_A(x) = 1 - (x-3)^2 \qquad 3 \le x \le 4.$$

The expected interval equals $EI(A) = \left[\frac{2}{3}, 3\frac{2}{3}\right]$. It is Chanas optimal interval I_0, too, because the left L and right R shape functions are the same. Let $J = [2-\sqrt{2}, 3+\sqrt{0.5}]$. Then we obtain $H(I_0, A) = 0{,}5926$ and $H(J, A) = 0{,}5858$. So, the Hamming distance for the Chanas interval I_0 is greater.

Now, we investigate the problem of approximation of the fuzzy number A by the trapezoidal number T. We propose one attempt based on the probabilistic interpretation of fuzzy numbers. We want to choose the trapezoidal fuzzy number $T = T(t_1, t_2, t_3, t_4)$ such that the induced random variables have the same expected value and variance, i.e.,

$$E(S_{A,1}) = E(S_{T,1}) = m_1, \quad E(S_{A,2}) = E(S_{T,2}) = m_2,$$

$$V(S_{A,1}) = V(S_{T,1}) = \sigma_1^2 \quad V(S_{A,2}) = V(S_{T,2}) = \sigma_2^2.$$

The sides of the trapezoidal fuzzy number are linear, then the random variables $S_{T,i}$ are uniformly distributed on $[t_1, t_2]$ and $[t_3, t_4]$. For the left sides of the fuzzy number A we obtain the following equations:

$$\begin{cases} t_1 + t_2 = 2m_1 \\ (t_2 - t_1)^2 = 12\sigma_1^2 \end{cases}.$$

Example 6. Let $f_A(x) = 2x - x^2$, where $0 \le x \le 1$, then $m_1 = 1/3$ and $\sigma_1^2 = 1/18$. We obtain the following solutions of the above equations:

$$t_1 = \frac{1}{3} - \sqrt{\frac{1}{6}} \approx -0.075 \qquad t_2 = \frac{1}{3} + \sqrt{\frac{1}{6}} \approx 0.742.$$

In some situations, such a trapezoidal fuzzy number may not exist. For instance, for triangular fuzzy numbers with convex sides.

Example 7. Let $f_A(x) = x^2$, where $0 \le x \le 1$ and $g_A(x) = (x - 2)^2$, where $1 \le x \le 2$. Then we obtain

$$t_1 = \frac{2}{3} - \sqrt{\frac{1}{6}}, \qquad t_2 = \frac{2}{3} + \sqrt{\frac{1}{6}} \approx 1.075,$$

$$t_3 = \frac{4}{3} - \sqrt{\frac{1}{6}} \approx 0.925, \qquad t_2 = \frac{4}{3} + \sqrt{\frac{1}{6}}$$

and $t_2 > t_3$ in this case. There is no such trapezoidal fuzzy number.

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Probabilistyczna interpretacja liczb rozmytych

Praca dotyczy probabilistycznej interpretacji liczb rozmytych. Badane są zależności zachodzące między rozmytymi liczbami a zbiorami losowymi. Stopień przynależności $\mu_A(x)$ jest interpretowany jako prawdopodobieństwo, że element x należy do przedziałowego zbioru losowego indukowanego przez poziomy rozmytego zbioru A. Przedziałowy zbiór losowy może być interpretowany jako para zmiennych losowych, których rozkłady są generowane przez strony rozmytego zbioru A.

Powyższa interpretacja probabilistyczna jest wykorzystana do porządkowania rozmytych liczb na podstawie stochastycznej dominacji. Pokazano, że naturalny porządek między rozmytymi liczbami jest częścią wspólną dwóch dominacji stochastycznych zachodzących między stronami porównywanych rozmytych liczb. W pracy rozpatrywane są naturalne porządki n-tego stopnia oraz przypadek trapezoidalnych liczb rozmytych. Omówiono problem „defuzzyfikacji", oparty na reprezentacji funkcyjnej relacji preferencji. Wartość oczekiwana rozmytej liczby, kombinacja wypukła dolnej i górnej wartości oczekiwanej oraz kombinacja zniekształconych wartości oczekiwanych – są przykładami tych reprezentacji funkcyjnych. Inna metoda „defuzzyfikacji" oparta jest na losowej symulacji. Przedstawiony jest też problem aproksymacji rozmytej liczby przedziałem oraz trapezoidalną rozmytą liczbą. Oczekiwany przedział rozmytej liczby oraz trapezoidalna rozmyta liczba z tą samą wartością oczekiwaną i wariancją traktowane są jako przykłady tej aproksymacji.